Probability problems: team contest

May 2019

1. Let Z be a real random variable, the log Laplace transform of Z is the function taking values in $\mathbb{R} \cup \{\infty\}$ defined for $\lambda \in \mathbb{R}$ by:

$$\Psi_Z(\lambda) = \ln \mathrm{E}(\exp(\lambda Z)).$$

1-1 For Z a random variable taking value in a bounded interval I, derive twice Ψ_Z and identify, for all λ, this second derivative to the variance of a random variable Z_λ, which we precise its distribution.

1-2 If Z is a random variable taking value in a bounded interval I, show that the variance of Z is upper bounded by $|I|^2/4$.

1-3 If Z is a random variable taking value in a bounded interval I. Show that for all $\lambda \in \mathbb{R}^+$,

$$\Psi_Z(\lambda) \leq \frac{|I|^2 \lambda^2}{8}.$$

1-4 Let $Z_1, \ldots, Z_n$ be intendant random variables, such that Z_i take value in $[a_i, b_i]$; denote $Z = \sum Z_i$ and $\tilde{Z} = Z - \mathrm{E}(Z)$. Show that

$$\Psi_{\tilde{Z}}(\lambda) \leq \frac{\lambda^2}{8} \sum (b_i - a_i)^2.$$

1-5 With same notations as in 1-4, show that for all $\epsilon > 0$,

$$\mathbf{P}(|\tilde{Z}| \geq \epsilon) \leq 2 \exp(-\frac{2\epsilon^2}{\sum (b_i - a_i)^2}).$$

2. Let $\{X_n\}$ be independent indentically distributed random variables with finite mean, and $S_n = \sum_{i=1}^n X_i$ be the partial sum. Show that the sequence $\{S_n/n\}$ is a reverse martingale. This means that

$$\mathbb{E}\left[\frac{S_n}{n} \middle| \mathscr{F}_{n+1}\right] = \frac{S_{n+1}}{n+1},$$

where

$$\mathscr{F}_n = \sigma\{S_n, S_{n+1}, \cdots\}$$

is the smallest σ-algebra generated by the random variables S_k for $k \geq n$.

3. Let $n \geq 2$ be an integer, show that, a random vector $X \in \mathbb{R}^n$ having independent components and its distribution is $O(n)$ invariant i.f.f. $X \sim \mathcal{N}(0, \sigma^2)$.
(Distribution is $O(n)$ invariant means that for any orthogonal matrix, that is a rotation, $P \in O(n)$, X and PX have the same distribution.)